

Information Percolation: Some General Cases with an Application to Econophysics

BY ALAIN BÉLANGER* AND GASTON GIROUX†

Université de Sherbrooke

We describe, at the microscopic level, the dynamics of N interacting components where the probability is very small when N is large that a given component interact more than once, directly or indirectly, up to time t , with any other component. Due to this fact, we can consider, at the macroscopic level, the quadratic system of differential equations associated with the interaction and establish an explicit formula for the solution of this system. We moreover apply our results to some models of Econophysics.

1 Introduction

In their paper, Duffie-Sun (2012) (see also Duffie (2012)), the authors provide a mathematical foundation for independent random matching of a large population. Here we develop an approach, inspired by Kac (1956), where the random matching is instead asymptotically independent. To do so, we start with a sequence of dynamical sets of interacting components, one for each integer N . For these dynamical systems we can show that when N is large the probability is very small that a component has interacted more than once directly or indirectly up to time t with any other component. Thanks to this fundamental property, we can link the microscopic and macroscopic levels using results from the theory of continuous-time Markov chains.

In section 2, we describe these dynamics with their symmetric interaction kernels. We consider interactions involving m components, for $m \geq 2$, and we suppose that the intensities of these dynamics have an adequate dependence on N . Our techniques enable us to obtain an explicit formula for the associated quadratic system of differential equations. We thereby extend the results first obtained in Duffie-Manso (2007) and pursued in Duffie-Giroux-Manso (2010). We note that our formula is valid for any interaction kernel and it is more explicit than the one obtained for the particular kernel considered in the latter article. It enables us, in particular, to obtain new results for models of Econophysics.

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* Département de finance, Université de Sherbrooke,
Sherbrooke, Canada, J1K 2R1. E-mail: alain.a.belanger@usherbrooke.ca

† 410 rue de Vimy, apt. 1, Sherbrooke, Canada, J1J 3M9. E-mail: gasgiroux@hotmail.com

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The statement and the proof that our formula solves the system of differential equations are done in section 3. In section 4, we present our applications to Econophysics. While in section 5, we come back to the fundamental property of the system's dynamics and obtain several intermediary results leading to its solution.

2 The dynamics.

We suppose that all components take their values in a measurable space, (E, \mathcal{E}) , (one can think of $(\mathbb{R}^d, B(\mathbb{R}^d))$) and their interactions are given by a symmetric probability kernel Q on the product space $(E^m, \mathcal{E}^{\otimes m})$ for $m \geq 2$. That is: the function $Q(x_1, x_2, \dots, x_m; C_1 \times \dots \times C_m)$ is measurable in (x_1, x_2, \dots, x_m) ; is a probability measure in $(C_1 \times \dots \times C_m)$; and satisfies $Q(x_1, x_2, \dots, x_m; C_1 \times \dots \times C_m) = Q(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)}; C_{\sigma(1)} \times C_{\sigma(2)} \times \dots \times C_{\sigma(m)})$ for any permutation σ of $\{1, 2, \dots, m\}$.

For each integer N , we consider an interacting set of N components which interact by groups of m according to the kernel Q . The interactions occur at each jump of a Poisson process with intensity $\lambda \frac{N}{m}$. Groups are undistinguishable so each group has a probability of $\binom{N}{m}^{-1}$ of being involved in a given interaction.

We show, in section 5, the fundamental property that enables us to obtain the system's solution. In the next section, we describe the formula and show that it is indeed the system's solution.

3 The solution of the macroscopic system.

The kernel Q allows us to describe the macroscopic evolution of the system with an associated system of quadratic differential equations via the evolution of the law of a component. This probability law, denoted μ_t , evolves with time and is in fact the solution of the Cauchy problem:

$$\frac{d\mu_t}{dt} = \mu_t^{\circ m} - \mu_t; \mu_0 = \mu$$

where

$$\mu^{\circ m}(C) \triangleq \int_{\mathbb{R}^m} \mu(dx_1) \mu(dx_2) \dots \mu(dx_m) Q(x_1, x_2, \dots, x_m; C \times E^{m-1}) \text{ for } C \in \mathcal{E}.$$

The probability law $\mu^{\circ m}$ is the law of a component after the interaction of m i.i.d. components with law μ . We can think of it as the law at the root of the m -ary tree with only one interaction. We will look at all the trees representing

the interaction history of a component up to time t . So for a tree, A , with more than one interaction, we divide the tree in m subtrees at that last interaction and continue recursively up to time 0 to define $\mu^{\circ_m A}$. (Please see figure 1 of section 5 for an example of an interaction tree.) Let \mathbb{A}_n be the set of all trees with n interactions (a.k.a. nodes), each node producing m branches. If $A_n \in \mathbb{A}_n$, then $\mu^{\circ_m A_n}$ denotes the law obtained by iteration of μ°_m} through the successive node of the tree when we place the law μ on each leaf of A_n .

Now we will show that our Cauchy problem has a unique solution which can be expressed, by conditioning on the number of interactions up to time t , and then by the component's history. Such conditionings give us

$$\mu_t = \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n} \quad (1)$$

where $\#_m(n) = \prod_{k=1}^{n-1} ((m-1)k+1)$ is the number of trees with n nodes, taking into account their branching orders; and $p_n(t)$ is the probability of having n branchings up to time t .

Finally, in order to show that the countable convex sum (1) gives the solution, we need the following lemma which will be proved in section 5.

Lemma 1 $p_n(t) = \frac{\#_m(n)}{(m-1)^n n!} e^{-t} (1 - e^{-(m-1)t})^n$.

Theorem 2 *The convex combination,*

$$\mu_t = \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$$

is the solution of the Cauchy problem

$$\frac{d\mu_t}{dt} = \mu_t^{\circ_m} - \mu_t ; \mu_0 = \mu.$$

Proof. Since the countable convex sum $(*)$ is normally summable, we can differentiate μ_t term by term to obtain:

$$-\mu_t + e^{-mt} \sum_{n \geq 1} n(1 - e^{-(m-1)t})^{n-1} \frac{1}{(m-1)^{n-1} (n-1)!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$$

Thus we need to show that:

$$\mu_t^{\circ_m}(C) = e^{-mt} \sum_{n \geq 0} n(1 - e^{-(m-1)t})^{n-1} \frac{1}{(m-1)^{n-1} n!} \sum_{A_{n+1} \in \mathbb{A}_{n+1}} \mu^{\circ_m A_{n+1}}(C) \quad (2)$$

Starting with the definition (on page 3), we have that the LHS of (2) is equal

to

$$\begin{aligned} & \int_{\mathbb{R}^m} \left(\sum_{i_1 \geq 0} e^{-t} (1 - e^{-(m-1)t})^{i_1} \frac{1}{(m-1)^{i_1} i_1!} \sum_{A_{i_1} \in \mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) \dots \\ & \dots \left(\sum_{i_m \geq 0} e^{-t} (1 - e^{-(m-1)t})^{i_m} \frac{1}{(m-1)^{i_m} i_m!} \sum_{A_{i_m} \in \mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) \dots \\ & \dots Q(x_1, \dots, x_m; C \times E^{m-1}) \end{aligned}$$

which is equal to

$$\begin{aligned} & \int_{\mathbb{R}^m} e^{-mt} \left\{ \sum_{n \geq 0} (1 - e^{-(m-1)t})^n \sum_{i_1 + \dots + i_m = n} \frac{1}{(m-1)^n i_1! \dots i_m!} \dots \right. \\ & \left. \left(\sum_{A_{i_1} \in \mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) \dots \left(\sum_{A_{i_m} \in \mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) \right\} Q(x_1, \dots, x_m; C \times E^{m-1}) \end{aligned}$$

which in turn is equal to

$$\int_{\mathbb{R}^m} e^{-mt} \left(\sum_{n \geq 0} (1 - e^{-(m-1)t})^n \frac{1}{(m-1)^n n!} F(i_1, \dots, i_m, n, \mu, A_{i_1}, \dots, A_{i_m}, Q, C) \right)$$

where

$$F(i_1, \dots, i_m, n, \mu, A_{i_1}, \dots, A_{i_m}, Q, C) = \dots$$

$$\begin{aligned} & \sum_{i_1 + \dots + i_m = n} \binom{n}{i_1} \binom{n-i_1}{i_2} \dots \binom{i_{m-1} + i_m}{i_{m-1}} \left(\sum_{A_{i_1} \in \mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) \dots \\ & \dots \left(\sum_{A_{i_m} \in \mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) Q(x_1, \dots, x_m; C \times E^{m-1}) \end{aligned}$$

And this last expression is a decomposition of the trees $A_{n+1} \in \mathbb{A}_{n+1}$ appearing in the RHS of (2) in m subtrees after the first node (taking the branching order into account). The two expressions are therefore equal and this proves the theorem. ■

Remark 3 *This section brings a simplification to the special case studied in section 3 of Duffie-Giroux-Manso (2010) where the term $a_{(m-1)(n-1)+1}$ can now be given explicitly as $\frac{1}{(m-1)^{n-1}(n-1)!}$. It also suggests that some of the results of Duffie-Malamud-Manso (2009) can be extended to the case where information exchanges involve m agents.*

Remark 4 We call the law $\mu_t = \sum_{n \geq 0} e^{-t}(1 - e^{-(m-1)t})^n \frac{1}{(m-1)^n n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$ an extended Wild sum [11] and note that the convex combination we obtain for the case $m = 2$ is indeed the Wild sum, $\mu_t = \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n \frac{1}{n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$, now well-known in the statistical physics of gases since the work of Kac (1956) [6].

4 An application in Econophysics: kinetic models with random perturbations

In Ferland-Giroux (1991) the authors study a class of kinetic equations of Kac's type and they show, for binary collisions, a convergence to the invariant law at an exponential rate. Bassetti *et al* (2011) show a similar result using different methods. The convergence in Ferland-Giroux (1991) is obtained along a set of convenient test functions with a telescoping technique due to Trotter (1959) and with the use of a version of Wild sums obtained from judicious conditioning. There is therefore the possibility of extending the results of Bassetti-Ladelli-Toscani (2011) for $m \geq 2$. We do this extension here only in the context of Ferland-Giroux (1991). For a review of Econophysics, in the case $m = 2$, one can read, among others, Düring-Matthes-Toscani (2010) and Bassetti *et al* (2011) for more recent work.

Let μ denote a probability law on \mathbb{R} and let $S_m(\mu)$ denote the law of $H_1 X_1 + H_2 X_2 + \dots + H_m X_m$ where the random variables $\{X_i\}$ are independent and of law μ and the variables $\{H_i\}$ are independent of each other and of the X variables. Under very general conditions, Rösler (1992) has shown that the transformation S_m has a fixed point. We will suppose here that the H -variables have values in $[0, 1]$, that their mean is $\frac{1}{m}$ and that they are not Bernoulli. We then have $E[H_1 + H_2 + \dots + H_m] = 1$ and $S_m(\mu)$ has the same first moment as μ . If we suppose moreover that $E[H_1^2 + H_2^2 + \dots + H_m^2] < 1$ then Rösler(1992) gives us the existence of a fixed point, denoted γ . This fixed point has a second moment as soon as μ does. Our goal is to establish the following result.

Theorem 5 *If we suppose that μ has a finite second moment then we have that the law $\mu_t = \sum_{n \geq 0} e^{-t}(1 - e^{-(m-1)t})^n \frac{1}{(m-1)^n n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$ converges to γ at the exponential rate $\eta = 1 - E[H_1^2 + H_2^2 + \dots + H_m^2]$.*

Our proof is an extension of Ferland-Giroux (1991) which treats the case $m = 2$.

Proof. Let us first consider the tree with m leaves, denoted A_1 . On each one of its leaves put independent random variables of law μ . Assume that these variables interact at a node to give $H_1 X_1 + H_2 X_2 + \dots + H_m X_m$. Let us call $\mu^{\circ_m A_1}$, or more simply μ°_m} , the law of this variable. In a similar fashion, we can consider γ°_m} (which is γ since it is a fixed point). We will first consider the

differences $|\langle \mu^{\circ m}, f \rangle - \langle \gamma^{\circ m}, f \rangle|$ for each $f \in C_b^2$. One way to bound this difference is to use the telescoping technique of Trotter (1959) where we replace one by one (from the left say) the variables with law μ by variables with law γ . We then obtain a sum of m terms of the form $|\langle \gamma^{\circ k} \circ \mu^{\circ m-k}, f \rangle - \langle \gamma^{\circ k+1} \circ \mu^{\circ m-k-1}, f \rangle|$ which we will bound. In our particular models we can write down these expressions explicitly as

$$|E[f(H_1 Y_1 + H_2 Y_2 + \dots + H_k Y_k + H_{k+1} X_{k+1} + \dots + H_m X_m)] - E[f(H_1 Y_1 + H_2 Y_2 + \dots + H_k Y_k + H_{k+1} Y_{k+1} + H_{k+2} X_{k+2} + \dots + H_m X_m)]|$$

where $Y_i : i = 1, \dots, k+1$ have law γ and X_j , with $j = k+1, \dots, m$, have law μ . Let $R_k = H_1 Y_1 + H_2 Y_2 + \dots + H_k Y_k + H_{k+2} X_{k+2} + \dots + H_m X_m$. Then we have

$$f(R_k + H_{k+1} X_{k+1}) = f(R_k) + f'(R_k) H_{k+1} X_{k+1} + f''(R_k^*) (H_{k+1} X_{k+1})^2$$

and

$$f(R_k + H_{k+1} Y_{k+1}) = f(R_k) + f'(R_k) H_{k+1} Y_{k+1} + f''(R_k^{**}) (H_{k+1} Y_{k+1})^2.$$

Which in turn give us ■

$$|\langle \mu^{\circ m}, f \rangle - \langle \gamma^{\circ m}, f \rangle| \leq cE \left[\sum_{i=1}^m H_i^2 \right].$$

We now need to iterate the process according to the different trees. Let \hat{A}_n denote the set of leaves of the tree A_n . Then the contribution of a leaf $u \in \hat{A}_n$ through its interactions down to the bottom of the tree is a product of the variables H_i with $i = 1, \dots, m$. Let us denote this product by C_u . For the tree A_n , the result of its interactions through the bottom of the tree will therefore go from $\sum_{u \in \hat{A}_n} C_u X_u$, when all the variables put on leaves have law μ , to $\sum_{u \in \hat{A}_n} C_u Y_u$ when all the variables have law γ . Applying the same techniques as above, namely a (longer) telescoping and a Taylor series expansion, we get

$$|\langle \mu^{\circ m A_n}, f \rangle - \langle \gamma^{\circ m}, f \rangle| \leq cE \left[\sum_{u \in \hat{A}_n} C_u^2 \right]$$

Let $e_n = \frac{1}{(m-1)^n n!} \sum_{A_n \in \mathbb{A}_n} E \left[\sum_{u \in \hat{A}_n} C_u^2 \right]$. If we decompose A_n in its m subtrees

from its first node we can apply a similar reasoning to Ferland-Giroux (1991) in order to obtain that $e_n \leq cn^{a-1}$ where $a = \frac{1-\eta}{m-1}$ and η is such that $1 - \eta =$

$$E \left[\sum_{i=1}^m H_i^2 \right].$$

We now reformulate these assertions in propositions and give their proof.

Lemma 6 We have $e_n = \left(\frac{1}{n}\right) \left(\frac{1-\eta}{m-1}\right) (1+e_1+\dots+e_{n-1})$. Therefore $e_n \leq cn^{a-1}$ with $a = \frac{1-\eta}{m-1}$.

Proof. The decomposition of each tree A_n in m subtrees $\{A_n^i\}_{i=1}^m$ at the first node enables us to write e_n as a sum of m similar terms

$$f_i = \frac{1}{(m-1)^n n!} \sum_{A_n \in \mathbb{A}_n} E \left[\sum_{u \in \widehat{A}_n^i} C_u^2 \right]$$

where \widehat{A}_n^i is the set of leaves of the i^{th} subtree. It suffices to treat the case $i = 1$. Let us decompose \mathbb{A}_n , the set of trees with n nodes, by the number, k , of nodes of the subtree A_n^1 . There are $\binom{n-1}{k} (m-1)^{n-1-k} (n-1-k)!$ such trees. Indeed, since we need to take into account the order of appearance of these nodes we have $\binom{n-1}{k}$ choices for the appearances of A_n^1 's nodes. Then we have $(m-1)^{n-1-k}$ ways to divide the remaining nodes in the other $m-1$ trees and finally, there are $(n-1-k)!$ choices for the nodes' appearances. Note that this number simplifies to $\frac{(n-1)!}{k!} (m-1)^{n-1-k}$. If we denote by $\mathbb{A}_{n,k}$ the subset of \mathbb{A}_n formed by the trees A_n for which their subtree A_n^1 has k nodes, we can then express f_i as

$$\begin{aligned} f_i &= \frac{1}{(m-1)^n n!} \sum_{k=0}^{n-1} \sum_{A_n \in \mathbb{A}_{n,k}} E \left[\sum_{u \in \widehat{A}_n^i} C_u^2 \right] \\ &= \frac{E[H_i^2]}{(m-1)n} \sum_{k=0}^{n-1} \frac{1}{(m-1)^k k!} \sum_{A_k \in \mathbb{A}_{n,k}} E \left[\sum_{u \in \widehat{A}_k^i} C_u^2 \right] \\ &= \frac{E[H_i^2]}{(m-1)n} \sum_{k=0}^{n-1} e_k, \text{ with } e_0 = 1. \end{aligned}$$

This proves the first assertion. A similar calculation to the proof of lemma 3 in Ferland-Giroux (1991) gives us the second result. ■

Proposition 7 For all $f \in C_b^2$ we have $|\langle \mu_t, f \rangle - \langle \gamma, f \rangle| \leq ce^{-\eta t}$

Proof. Once again, it suffices to follow the proof of theorem 3 in Ferland-Giroux (1991) with the extended Wild sum and replacing $1-\eta$ by $\frac{1-\eta}{m-1}$. ■

5 The extended Wild sums.

5.1 When graphs become trees

In all our cases, we have an underlying market structure which is a Kac walk with interactions involving m agents. We add exponential times to obtain a marked

Poisson process whose marks are horizontal lines linking the agents participating in a given interaction. This enables us to describe the limit law of an agent, under an appropriate conditioning, as a countable convex combination on trees which is, as we have shown in section 3, the global solution of the associated differential equation on the space of probability laws.

Here we explain how we came to that convex combination. We start our study by an analysis of the dynamics of the intrinsic structure of the large set of interacting agents when the number of agents increases. We assume that each interaction involves m agents, $m \geq 2$. More specifically, we consider a set of N agents whose interactions happen at unexpected times so these interactions' occurrences follow a Poisson process. Since agents are interchangeable, each group has an equal probability of meeting of $\binom{N}{m}^{-1}$. If we suppose the intensity of the meetings to be $\frac{N}{m}$ then each agent has a meeting rate λ which can be assumed to equal 1 under a time change.

For N fixed and starting at time 0, we assign a vertical position to each agent. The down movement represents the passage of time, see figure 1 on page 9. Each time a group of agents interacts, we draw a horizontal line between those agents and we draw a vertical line at each agent's position connecting 0 to the horizontal line just drawn, so we see a random graph being formed. When we stop this graph at time t , we obtain the finite graph of all interactions that have taken place. Moreover, the history up to time t of a given agent, call it P , is described by the random sub-graph connecting all agents who have interacted directly or indirectly with P . A sample history of P 's meetings/interactions may look like figure 1 below.

The number of meetings is random but we can condition on it. The law of the finite graph is reversible since the meeting times are uniform on $[0, t]$. We want to show that a random graph representing the history of P can be replaced by a random tree as the number of agents, N , grows. If we look at figure 1, we see that the inclusion in the second meeting of one of the investors having participated in the first one, or the inclusion in the third meeting of an investor from the first or second one would create a cycle in our subgraph. As N grows though, the chance of meeting an investor previously encountered directly or indirectly tends to zero.

To see this, let us consider the subgraph of P 's history up to time t . Starting at time t , we pursue each one of the encountered vertical lines in P 's history backward in time until we reach the next horizontal line. If the inclusion of the horizontal line in our graph does not create a cycle (i.e. no pair of investors were involved directly or indirectly in a previous meeting) we include the line, if not we remove it. Proceeding in this fashion up to time 0 we get a tree with n branchings, say, which has the same law as the law of a tree obtained by a pure-birth process. Namely, the tree starting at P 's vertical line at time t with intensity 1 and which at time 0 has intensity $(m-1)n+1$ and that same number of leaves. Between two branchings of this process a graph representing P 's meeting history can have a random

Figure 1:

number of additional horizontal lines following a Poisson law of parameter at most $\frac{N}{m} \left(\binom{(m-1)n+1}{2} \binom{N}{m}^{-1} \right)$. We will now bound the expectation of these supplementary horizontal lines by a majorant which tends to 0 as N increases. Indeed, since the mean number of redundant lines when there are n branchings up to time t is at most $\frac{N}{m} \left(\binom{(m-1)n+1}{2} \binom{N}{m}^{-1} \right)$, we have that the mean number of redundant horizontal lines is bounded above by $\sum_{n \geq 0} \frac{N}{m} \left(\binom{(m-1)n+1}{2} \binom{N}{m}^{-1} \right) p_{N,n}(t)$, where $p_{N,n}(t)$ is the probability of having n branchings up to time t of the pure birth process with successive branching waiting times following exponential laws of parameter

$$\lambda_{N,n} = \frac{N}{m} ((m-1)n+1) \binom{N - ((m-1)n+1)}{m-1} \binom{N}{m}^{-1}$$

Since

$$\begin{aligned} & \frac{N}{m} ((m-1)n+1) \binom{N - ((m-1)n+1)}{m-1} \binom{N}{m}^{-1} \\ &= \frac{((m-1)n+1) \binom{N - ((m-1)n+1)}{m-1}}{\binom{N-1}{m-1}} \quad (3) \\ &\leq (m-1)n+1 \end{aligned}$$

then $p_{N,n}(t)$ is stochastically smaller than the law obtained with the intensities $\lambda_n = (m-1)n+1$, which in turn are less than the intensities $\bar{\lambda}_n = m(n+1)$. Its transition kernel is then obtained by solving Kolmogorov's affine system of equations:

$$\begin{aligned} \frac{d\bar{p}_t(0)}{dt} &= -\bar{p}_t(0) \\ \frac{d\bar{p}_t(n)}{dt} &= mn\bar{p}_t(n-1) - m(n+1)\bar{p}_t(n) \quad ; n \geq 1. \end{aligned}$$

Thus the latter intensities give us a geometric law $\bar{p}_t(n) = e^{-mt}(1 - e^{-mt})^n$. Since geometric laws have finite moments of all orders, the mean number of redundant horizontal lines is bounded above by a quantity converging to 0.

For more details on Kolmogorov systems of equations for pure birth processes we refer the reader to Lefebvre (2006), for instance.

Thus, after having specified the initial agents' states and their interaction kernels, we can approximate P 's law using the tree obtained from removing all redundant horizontal lines from its graph. We will use this fact in the next sub-section.

5.2 Limit countable convex combination

We will now show that these random trees whose branching intensities depend on N can be approximated by trees with branching intensities independent of N . Taking into account that P 's tree history is random with intensities depending on N , we could write P 's law, denoted by $\mu_t^{*,N}$, with complex formulae depending on N . Since our markets have a large number of investors, it is preferable instead to work with the limit of these laws. We note from (3) above that $\lambda_{N,n} \rightarrow ((m-1)n+1)$.

Let $p_n(t)$ ($\triangleq p_t(n)$) be the solution of the affine Kolmogorov system of equations:

$$\begin{aligned} \frac{dp_t(0)}{dt} &= -p_t(0) \\ \frac{dp_t(n)}{dt} &= ((m-1)(n-1)+1)p_t(n-1) - ((m-1)n+1)p_t(n) \quad ; n \geq 1. \end{aligned} \quad (4)$$

Recall from the first section that $\mu_t = \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$.

Proposition 8 *The sequence of laws $\mu_t^{*,N}$ converges to μ_t as N increases.*

Proof. By Kurtz (1969) we have that $p_{N,n}(t) \rightarrow p_n(t)$ as N increases. But $(p_n(t))_{n \geq 0}$ is a probability law, so for $\epsilon > 0$, there exists $n(\epsilon)$ such that

$\sum_{n \geq n(\epsilon)} p_n(t) < \epsilon$. Now let $N(\epsilon)$ be such that $N > N(\epsilon)$ implies that $|p_{N,n}(t) - p_n(t)| < \frac{\epsilon}{n(\epsilon)}$ for $0 \leq n \leq n(\epsilon)$. We then have for $C \in \mathcal{E}$ and $N > N(\epsilon)$

$$|\mu_t^{*,N}(C) - \mu_t(C)| \leq \sum_{n=0}^{n(\epsilon)} |p_{N,n}(t) - p_n(t)| + 2\epsilon \leq 3\epsilon$$

since $\frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}(C) \leq 1$ and $(p_{N,n}(t))_{n \geq 0}$ are probability laws. Our claim is proved. ■

Lemma 9 $p_t(n) = \frac{\#_m(n)}{(m-1)^n n!} e^{-t} (1 - e^{-(m-1)t})^n$

Proof. We need to solve the affine Kolmogorov system of equations (4).

Proceeding by induction we have:

$$\begin{aligned}\frac{dp_t(0)}{dt} &= e^{-t} \\ \frac{dp_t(n)}{dt} &= ((m-1)(n-1)+1)e^{-(n(m-1)+1)t} \int_0^t e^{(n(m-1)+1)s} p_s(n-1) ds\end{aligned}$$

To prove the lemma it suffices to note that $\#_m(n) = \#_m(n-1)((n-1)(m-1)+1)$ and that $e^{(n(m-1)+1)s} e^{-s}(1 - e^{-(m-1)s})^{n-1} = e^{(m-1)s}(e^{(m-1)s} - 1)^{n-1}$ is the derivative of $\frac{1}{(m-1)^n}(e^{(m-1)s} - 1)^n$. ■

And this shows that the limit law of P is indeed the extended Wild sum which we have shown to be the solution of the ODE associated to the interacting system .

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Alain Bélanger,
 Département de finance,
 Université de Sherbrooke,
 2 500 boul. de l'Université,
 Sherbrooke, Canada, J1K 2R1
 E-address: alain.a.belanger@usherbrooke.ca

Gaston Giroux,
 410 rue de Vimy, apt. 1,
 Sherbrooke, Canada, J1J 3M9
 E-address: gasgiroux@hotmail.com

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